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M. Ademollo and R. Gatto: METHODS FOR DETERMINING THE SPIN OF Ξ

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ABSTRACT.

Tests of Ξ spin are proposed which consist in measuring and comparing among each other some averages over the angular distribution of the Λ resulting from Ξ decay. The averages can directly be constructed out of experimental data and a discussion on the statistical errors arising from application of the tests to a finite sample of Ξ-decays is given. General statements arising from a quadratic relation among the coefficients of the production density matrix valid for production on unpolarized nucleons are also given. The question of the Ξ parity is briefly considered assuming only s and p waves at production.

1 - INTRODUCTION - GENERAL FORMULAS.

1.1 - No conclusive evidence has been obtained so far on the Ξ hyperon spin[1]. We report here briefly on some possible methods for its determination.

The spin tests that we propose consist in measuring and comparing among each other some averages over the angular distribution of the Λ resulting from Ξ decay. The averages are defined in section 2 and in section 3. In particular in sections 2.4 and 3.2 we illustrate possible ways of comparison of the averages to determine the Ξ spin.

The construction of the averages directly from the experimental data is explained in sections 2.1 and 4. Also, in section 4, the important question of minimizing the statistical error in a finite sample of Ξ decays is discussed.

More general statements on the Ξ spin are made in sections 5.2 and 5.3 as a consequence of a general quadratic relation among the coefficients of the production density matrix, which is valid for production on unpolarized nucleons.

The comparison with the Lee-Yang tests is examined in section 5.1 and production under Adair's conditions is considered in section 5.4.

The possibility of having only s and p waves in the production process is briefly considered in section 6.

1.2 - Let us consider the chain of reactions

$$K^- + p \longrightarrow \Xi^- + K^+ \tag{1}$$

$$\Xi^- \longrightarrow \Lambda + \pi^- \tag{2}$$

we call $\vec{u}$ and $\vec{u}'$ respectively the unit vectors along the proton momentum and the Ξ^- momentum in the center of mass frame for reaction (1); $\vec{n} = \dfrac{\vec{u} \times \vec{u}'}{|\vec{u} \times \vec{u}'|}$ the normal to the production plane; $\vec{v}$ the unit vector along the Λ momentum in the center of mass frame for reaction (2); and we define $\vec{w} = \vec{n} \times \vec{u}$.

If the Ξ spin is $\frac{3}{2}$ its density matrix, after its production according to (1), for unpolarized initial nucleons, can be written as

$$\begin{aligned} \mathcal{U}(\theta) = \tfrac{1}{4} &+ b_1(\theta)\, T(\vec{n}) + c_1(\theta)\, T(\vec{u},\vec{u}) + c_2(\theta)\, T(\vec{w},\vec{w}) + c_3(\theta)\, T(\vec{u},\vec{w}) + \\ &+ d_1(\theta)\, T(\vec{n},\vec{n},\vec{n}) + d_2(\theta)\, T(\vec{n},\vec{u},\vec{u}) + d_3(\theta)\, T(\vec{n},\vec{u},\vec{w}) \end{aligned} \tag{3}$$

where $b_1(\theta)$ $c_i(\theta)$ and $d_i(\theta)$ are real functions of the production angle θ. The quantities T are spin operators defined as in reference (2) (see also reference (3) footnote n.6).

The coefficients b_1, c_i, and d_i are not completely unknown. First, they must satisfy inequalities expressing the fact that the probabilities for finding the Ξ with given spin component must be positive. These probabilities are the diagonal elements of the density matrix U . Furthermore, we will see that a quadratic condition must be statisfied among the coefficients b, c, and d, deriving from the fact that the density matrix must be expressible as an incoherent mixture of two pure spin states, corresponding to the two possible orientations of the initial nucleon spin in the production process. We shall make use of such limitations in section 5.

1.3 - The angular distribution I ($\vec{v}$) (normalized to unity in the whole solid angle) of the Λ emitted in the subsequent Ξ decay is given by:

$$
\begin{aligned}
4\pi I(\vec{v}) = {} & 1 + \frac{2}{\sqrt{5}} \alpha b_1(\theta)(\vec{n}\cdot\vec{v}) - c_1(\theta)[3(\vec{u}\cdot\vec{v})^2-1] - c_2(\theta)[3(w\cdot v)^2-1] - \\
& - 3c_3(\theta)(\vec{u}\cdot\vec{v})(\vec{w}\cdot\vec{v}) - \frac{3}{\sqrt{5}}\alpha\Big\{d_1(\theta)[5(\vec{n}\cdot\vec{v})^3-3(\vec{n}\cdot\vec{v})] + \\
& + d_2(\theta)[5(\vec{n}\cdot\vec{v})(\vec{u}\cdot\vec{v})^2-(n\cdot v)] + 5d_3(\theta)(\vec{n}\cdot\vec{v})(\vec{u}\cdot\vec{v})(\vec{w}\cdot\vec{v})\Big\}
\end{aligned}
\tag{4}
$$

The polarization $\vec{P}$ of the emitted Λ is:

$$
\begin{aligned}
4\pi I(\vec{v})\vec{P} = {} & \alpha\vec{v} + \frac{2}{\sqrt{5}} b_1(\theta)\big[(\vec{n}\cdot\vec{v})\vec{v} + 2\epsilon\,\vec{v}\times(\vec{n}\times\vec{v}) - 2\beta(\vec{n}\times\vec{v})\big] - \\
& - \alpha\vec{v}\Big\{c_1(\theta)[3(\vec{u}\cdot\vec{v})^2-1] + c_2(\theta)[3(\vec{w}\cdot\vec{v})^2-1] + 3c_3(\theta)(\vec{u}\cdot\vec{v})(\vec{w}\cdot\vec{v})\Big\} - \\
& - \frac{3}{\sqrt{5}} d_1(\theta)\Big\{[5(\vec{n}\cdot\vec{v})^3-3(\vec{n}\cdot\vec{v})]\vec{v} + [5(\vec{n}\cdot\vec{v})^2-1][\varepsilon\,\vec{v}\times(\vec{n}\times\vec{v}) - \beta(\vec{n}\times\vec{v})]\Big\} - \\
& - \frac{1}{\sqrt{5}} d_2(\theta)\Big\{[5(\vec{u}\cdot\vec{v})^2-3](\vec{n}\cdot\vec{v})\vec{v} + 10(\vec{u}\cdot\vec{v})^2(\vec{n}\cdot\vec{v})\vec{v} + \\
& + \varepsilon\big[[5(u\cdot v)^2-1]\vec{v}\times(\vec{n}\times\vec{v}) + 10(\vec{u}\cdot\vec{v})(\vec{n}\cdot\vec{v})\vec{v}\times(\vec{u}\times\vec{v})\big] - \\
& - \beta\big[[5(u\cdot v)^2-1](\vec{n}\times\vec{v}) + 10(\vec{u}\cdot\vec{v})(\vec{n}\cdot\vec{v})(\vec{u}\times\vec{v})\big]\Big\} - \\
& - \sqrt{5}\, d_3(\theta)\Big\{3(\vec{u}\cdot\vec{v})(\vec{n}\cdot\vec{v})(\vec{w}\cdot\vec{v})\vec{v} + \varepsilon\big[(\vec{u}\cdot\vec{v})(\vec{w}\cdot\vec{v})\vec{v}\times(\vec{n}\times\vec{v}) + \\
& + (\vec{n}\cdot\vec{v})(\vec{w}\cdot\vec{v})\vec{v}\times(\vec{u}\times\vec{v}) + (\vec{n}\cdot\vec{v})(\vec{u}\cdot\vec{v})\vec{v}\times(\vec{w}\times\vec{v})\big] - \\
& - \beta\big[(\vec{u}\cdot\vec{v})(\vec{w}\cdot\vec{v})(\vec{n}\times\vec{v}) + (\vec{n}\cdot\vec{v})(\vec{w}\cdot\vec{v})(\vec{u}\times\vec{v}) + (\vec{n}\cdot\vec{v})(\vec{u}\cdot\vec{v})(\vec{w}\times\vec{v})\big]\Big\}
\end{aligned}
\tag{5}
$$

Eqs. (4) and (5) completely determine the density matrix of the Λ produced trough the chain of reactions (1) and (2), when reaction (1) occurs on unpolarized protons. The parameters α, β and ϵ are given in terms of the elements of the Ξ decay matrix, R_1 for decay into final $\Lambda - \pi$ p - wave, and R_2 for decay into final $\Lambda - \pi$ d - wave:

(6) $$\alpha = \frac{2\,\mathrm{Re}(R_1 R_2^*)}{|R_1|^2 + |R_2|^2} = \frac{2|R_1||R_2|}{|R_1|^2+|R_2|^2}\cos(\delta_1 - \delta_2)$$

(6') $$\beta = \frac{2\,\mathrm{Im}(R_1 R_2^*)}{|R_1|^2 + |R_2|^2} = \frac{2|R_1||R_2|}{|R_1|^2+|R_2|^2}\sin(\delta_1 - \delta_2)$$

(6") $$\epsilon = \frac{|R_1|^2 - |R_2|^2}{|R_1|^2+|R_2|^2}$$

Note that

(6"') $$\alpha^2 + \beta^2 + \epsilon^2 = 1$$

The real numbers δ_1 and δ_2 are the Λ-π scattering phase-shifts for p-wave and d-wave respectively, at an energy equal to the Q-value of Ξ decay. In order to introduce explicitly the phase shifts δ in Eqs.(6) we have made use of time reversal invariance.

From the expressions (4) and (5) for the angular distribution and polarization of Λ one can derive many different criteria to check the hypotesis of spin $\frac{3}{2}$ and, by comparing with analogous expressions for spin $\frac{1}{2}$, to distinguish between these two possibilities.

1.4 - The expressions for the angular distribution $I(\vec{\nu})$ and for the polarization P of the emitted Λ for Ξ spin $\frac{1}{2}$ are, as well known:

(7) $$4\pi I(\vec{\nu}) = 1 + \alpha P_0 (\vec{n}\cdot\vec{\nu})$$

(8) $$4\pi I(\vec{\nu})\vec{P} = \alpha\vec{\nu} - \beta P_0(\vec{n}\times\vec{\nu}) + P_0(\vec{n}\cdot\vec{\nu})\vec{\nu} - \epsilon P_0 \vec{\nu}\times(\vec{n}\times\vec{\nu})$$

Here P_o is the component of the Ξ polarization along n and the parameters α, β and ϵ are given by:

$$\alpha = \frac{2 Re(R_o R_1^*)}{|R_o|^2 + |R_1|^2} = \frac{2|R_o||R_1|}{|R_o|^2+|R_1|^2} \cos(\delta_o - \delta_1) \tag{9}$$

$$\beta = \frac{2 Im(R_o R_1^*)}{|R_o|^2 + |R_1|^2} = \frac{2|R_o||R_1|}{|R_o|^2+|R_1|^2} \sin(\delta_o - \delta_1) \tag{9'}$$

$$\epsilon = \frac{|R_1|^2 - |R_o|^2}{|R_o|^2+|R_1|^2} \tag{9"}$$

We have made use of time-reversal invariance to introduce the s-wave and p-wave Λ-π phase shifts δ_o and δ_1 respectively. Again $\alpha^2 + \beta^2 + \epsilon^2 = 1$.

2 - POLARIZATION ANALYSIS.

2.1 - Let us first derive tests from measurements of the Λ polarization.

The Λ polarization is measured directly through the observation of its decay asymmetries. We are interested in the components of Λ polarization along three ortogonal axes whose unit vectors we call $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$. We take:

$$\vec{v}_1 = \vec{v}\,; \qquad \vec{v}_2 = \frac{\vec{n} \times \vec{v}}{|\vec{n} \times \vec{v}|}\,; \qquad \vec{v}_3 = \frac{\vec{v} \times (\vec{n} \times \vec{v})}{|\vec{v} \times (\vec{n} \times \vec{v})|} \tag{10}$$

The component of the Λ polarization along $\vec{v}_i$ is essentially measured by the difference between the number of decay events with the pion (from Λ decay) going forward with respect to the plane normal to $\vec{v}_i$ and the number of events with the pion going backward .

More precisely for a sample of Λ at rest with polarization P we have

$$\vec{P} \cdot \vec{v}_i = \frac{2}{\alpha_\Lambda} \frac{N_i^+ - N_i^-}{N_i^+ + N_i^-} \tag{11}$$

where α_Λ is the Λ decay parameter, N_i^+ is the number of decays with the emitted pion going forward (i.e. forming an angle less than 90° with respect to $\vec{v}_i$) and N_i^- is the number of decays with the emitted pion going backward (i.e. forming an angle larger than 90° with respect to $\vec{v}_i$).

We consider averages of the components of the Λ polarization over the Λ direction of flight, weighted by suitable weight functions. We call Θ and Φ the polar coordinates of $\vec{v}$ in a system where $\vec{n}$ is the polar axis and $\vec{u}$ is the x axis. We define the average of a quantity A as

(12) $$\langle A \rangle = \iint d\Phi \, d(-\cos\Theta)\, I\, A$$

2.2 - For Ξ spin $\frac{3}{2}$, from (5), (10) and (12) we obtain:

(13a) $$\langle \vec{P}\cdot\vec{V}_1 \, Y_1^0(\theta,\Phi)\rangle = b_1(\theta)\frac{1}{\sqrt{15\pi}}$$

(13b) $$\langle \vec{P}\cdot\vec{V}_1 \, Y_3^0(\theta,\Phi)\rangle = -\left[d_1(\theta)-\tfrac{1}{2}d_2(\theta)\right]\frac{3}{\sqrt{35\pi}}$$

(13c) $$\langle \vec{P}\cdot\vec{V}_1 \, Y_3^{\pm 2}(\theta,\Phi)\rangle = -\left[d_2(\theta)\pm i d_3(\theta)\right]\sqrt{\frac{3}{56\pi}}$$

13d) $$\langle \vec{P}\cdot\vec{V}_2 \, Y_3^{\pm 2}(\theta,\Phi)\rangle = \pm i\left[d_2(\theta)\pm i d_3(\theta)\right]\epsilon\,\frac{5}{256}\sqrt{\frac{21\pi}{2}}$$

(13e) $$\langle \vec{P}\cdot\vec{V}_3 \, \sin\theta\rangle = b_1(\theta)\,\epsilon\,\frac{8}{3\sqrt{5}}$$

(13f) $$\langle \vec{P}\cdot\vec{V}_3 \,(4\cos^2\theta-1)\rangle = -\left[d_1(\theta)-\tfrac{1}{2}d_2(\theta)\right]\epsilon\,\frac{3\pi\sqrt{5}}{16}$$

(13g) $$\langle \vec{P}\cdot\vec{V}_3 \, Y_2^{\pm 2}(\theta,\Phi)\rangle = -\left[d_2(\theta)\pm i d_3(\theta)\right]\epsilon\,\frac{15}{512}\sqrt{\frac{3\pi}{2}}$$

14a) $$\langle \vec{P}\cdot\vec{V}_1 \rangle = \alpha$$

(14b) $$\langle \vec{P}\cdot\vec{V}_1 \, Y_2^0(\theta,\Phi)\rangle = \left[c_1(\theta)+c_2(\theta)\right]\alpha\,\frac{1}{\sqrt{20\pi}}$$

(14c) $$\langle \vec{P}\cdot\vec{V}_1\, Y_2^{\pm 2}(\theta,\Phi)\rangle = -[c_1(\theta) - c_2(\theta) \pm i c_3(\theta)]\alpha \frac{1}{4}\sqrt{\frac{6}{5\pi}}$$

(14d) $$\langle \vec{P}\cdot\vec{V}_2 \sin\theta\rangle = -b_1(\theta)\beta \frac{8}{3\sqrt{5}}$$

(14e) $$\langle \vec{P}\cdot\vec{V}_2 (4\cos^2\theta - 1)\rangle = [d_1(\theta) - \tfrac{1}{2} d_2(\theta)]\beta \frac{3\pi\sqrt{5}}{16}$$

(14f) $$\langle \vec{P}\cdot\vec{V}_2\, Y_2^{\pm 2}(\theta,\Phi)\rangle = [d_2(\theta) \pm i d_3(\theta)]\beta \frac{15}{512}\sqrt{\frac{3\pi}{2}}$$

(14g) $$\langle \vec{P}\cdot\vec{V}_3\, Y_3^{\pm 2}(\theta,\Phi)\rangle = \pm i[d_2(\theta) \pm i d_3(\theta)]\beta \frac{5}{256}\sqrt{\frac{21\pi}{2}}$$

The relations (13) and (14) generalize to the case of parity non-conserving decay the analogous relations of Gatto and Stapp[3] (compare with their equations (5), (6) and (7)). For a parity conserving decay $\alpha = \beta = 0$ and $\epsilon = \pm 1$ according to the relative intrinsic parities of the particles ($\epsilon = +1$ for final p-wave and $\epsilon = -1$ for final d-wave).

We also note that if the Λ polarization $\vec{P}$ is split into a sum of a vector and of a pseudovector, only the pseudovector contributes to the averages (13), whereas only the vector contributes to the averages (14) (all proportional to α or β).

2.3 - In a similar way, for Ξ spin $\frac{1}{2}$ we find from (8), (10) and (12):

(15a) $$\langle \vec{P}\cdot\vec{V}_1\, Y_1^{0}(\theta,\Phi)\rangle = P_0(\theta)\frac{1}{\sqrt{12\pi}}$$

(15e) $$\langle \vec{P}\cdot\vec{V}_3 \sin\theta\rangle = -P_0(\theta)\epsilon\frac{2}{3}$$

(16a) $$\langle \vec{P}\cdot\vec{V}_1\rangle = \alpha$$

(16d) $$\langle \vec{P}\cdot V_2 \sin\theta\rangle = -P_0(\theta)\beta\frac{2}{3}$$

while all the remaining averages are zero:

$$(15') \qquad \langle \vec{P}\cdot\vec{V}_1 Y_3^0 \rangle = \langle \vec{P}\cdot\vec{V}_1 Y_3^{\pm 2} \rangle = \langle \vec{P}\cdot\vec{V}_2 Y_3^{\pm 2} \rangle = \langle P\cdot V_3 (4\cos^2\theta - 1) \rangle = \langle \vec{P}\cdot\vec{V}_3 Y_2^{\pm 2} \rangle = 0$$

$$(16') \qquad \langle \vec{P}\cdot\vec{V}_1 Y_2^0 \rangle = \langle \vec{P}\cdot\vec{V}_1 Y_2^{\pm 2} \rangle = \langle \vec{P}\cdot V_2 (4\cos^2\theta - 1) \rangle = \langle \vec{P}\cdot\vec{V}_2 Y_2^{\pm 2} \rangle = \langle \vec{P}\cdot\vec{V}_3 Y_3^{\pm 2} \rangle = 0$$

2.4 - There are many ways in which the above relations can be used to determine the spin of Ξ . A general argument is the following: if any of the averages (15') and (16') turns out to be different from zero, the spin cannot be $\frac{1}{2}$; on the other hand, if they are all zero and $\alpha \neq 0$ we can conclude, as we shall see later, that the spin must be $\frac{1}{2}$.

For spin $\frac{1}{2}$ we can obtain the Ξ polarization P_0 from (15a) and the decay parameters α, β and ϵ from (16a), (16d) and (15e) afther eliminating P_0; the sum of their squares must be 1.

To test for spin $\frac{3}{2}$ one can for instance derive b_1 and ε from (13a) and (13e); the averages (13b), (13c), (13d), (13f) and (13g) must then all be expressible through the three real numbers d_1, d_2 and d_3. Furthermore, after obtaining β from (14d), the averages (14e), (14f) and (14g) must also be expressible through d_1, d_2 and d_3. The parameter α can be evaluated from (14a), and from (14b) and (14c) one can obtain c_1, c_2 and c_3.

3 - ANGULAR DISTRIBUTION ANALYSIS.

3.1 - Also from the Λ angular distribution we can obtain averages analogous to that considered above.

For Ξ spin $\frac{3}{2}$, from (4) and (12) we find

$$(17a) \qquad \langle Y_2^0(\theta,\Phi) \rangle = [c_1(\theta) + c_2(\theta)] \frac{1}{\sqrt{20\pi}}$$

$$(17b) \qquad \langle Y_2^{\pm 2}(\theta,\Phi) \rangle = -[c_1(\theta) - c_2(\theta) \pm i c_3(\theta)] \frac{3}{2\sqrt{30\pi}}$$

$$(18a) \qquad \langle Y_1^0(\theta,\Phi) \rangle = b_1(\theta)\, \alpha \frac{1}{\sqrt{15\pi}}$$

(18b) $$\langle Y_3^0(\theta,\Phi)\rangle = -\left[d_1(\theta) - \tfrac{1}{2}d_2(\theta)\right]\alpha\,\frac{3}{\sqrt{35\pi}}$$

(18c) $$\langle Y_3^{\pm 2}(\theta,\Phi)\rangle = -\left[d_2(\theta) \pm i d_3(\theta)\right]\alpha\sqrt{\frac{3}{56\pi}}$$

where as for Ξ spin $\frac{1}{2}$, from (7) we obtain:

(19a) $$\langle Y_1^0(\theta,\Phi)\rangle = P_0(\theta)\,\alpha\,\frac{1}{\sqrt{12\pi}}$$

(20') $$\langle Y_2^0\rangle = \langle Y_2^{\pm 2}\rangle = 0$$

(19') $$\langle Y_3^0\rangle = \langle Y_3^{\pm 2}\rangle = 0$$

3.2 - These relations can also be used to determine the spin of Ξ, alone or, much more usefully, in combination with (13) and (14).

If they are used alone, the general argument to check the spin of Ξ is analogous to that seen before: if any of the averages (19') and (20') turn out to be different from zero, the spin cannot be $\frac{1}{2}$; if, on the other hand, they are all zero and we know, e.g. from (14a), that $\alpha \neq 0$, the spin must be $\frac{1}{2}$.

If however we find that not only (19') and (20') but also (15') and (16') are zero (actually it would be sufficient that (13b), (13c), (17a) and (17c) were zero) we could conclude that the spin is $\frac{1}{2}$, without the condition $\alpha \neq 0$. In fact we shall see that if the spin is $\frac{3}{2}$ it is impossible that all the coefficients c_i and d_i are simultaneously vanishing.

To test for spin $\frac{3}{2}$, we notice that (17) and (18) are hardly sufficient to determine the coefficients b_1, c_i and d_i, provided we know for example α. These determinations are however independent from (13) and (14) and must agree with them.

4 - APPLICATION TO FINITE SAMPLES.

4.1 - From the experimental point of view, when the number of events is not very large the statistical error on the averages may be important. We shall briefly see that, if we are not interested in a very detailed analysis, we can put together all the events at any production angle to reduce the statistical error.

The averages of the polarization that we have considered are of the form

$$m(\theta) = \langle \vec{P}\cdot\vec{V}_i \; F(\theta,\Phi)\rangle$$

Experimentally they are approximated by:

$$\bar{m}(\theta) = \frac{2}{\alpha_\Lambda}\,\frac{1}{N(\theta)}\sum_{(\theta)}{}_j{}' \; \varepsilon_j^i F_j \tag{21}$$

where use has been made of (11); $F_j = F(\theta_j,\Phi_j)$ is the value of the weight function $F(\theta,\Phi)$ at the polar coordinates of the Λ relative to the event j; $\varepsilon_j^i = \pm 1$ according to the case that the pion emitted in the Λ decay forms an angle $\lessgtr 90°$ with the direction $\vec{V}_i$, and the sum is performed over all the $N(\theta)$ events whose production angle lies in a certain interval around θ .

The order of magnitude of the error on $m(\theta)$ is (we suppose for simplicity F and m real):

$$\frac{\bar{m}(\theta)-m(\theta)}{m(\theta)} \simeq \frac{1}{\sqrt{N(\theta)}}\,\frac{\delta(\theta)}{m(\theta)}$$

where

$$\delta^2(\theta) = \left(\frac{2}{\alpha_\Lambda}\right)^2 \frac{1}{N(\theta)}\sum_{(\theta)}{}_j{}' \; F_j^2 - \bar{m}^2(\theta)$$

Let us define

$$M = 2\pi\int_{-1}^{1} I_0(\theta)\, m(\theta)\, f(\theta)\, d\cos\theta$$

where $I_0(\theta)$ is the production angular distribution (normalized to unity in the whole solid angle) and $f(\theta)$ is a suitable weight function.

Experimentally M is approximed by:

$$\bar{M} = \frac{2}{\alpha_\Lambda}\,\frac{1}{N}\sum_r \varepsilon_r^i F_r f_r \tag{22}$$

where now the sum is performed over all the N events.

The error on M is of the order.

$$\frac{\bar{M}-M}{M} \simeq \frac{1}{\sqrt{N}} \frac{\Delta}{M}$$

where

$$\Delta^2 = \left(\frac{2}{\alpha_\Lambda}\right)^2 \frac{1}{N} \Sigma_r F_r^2 f_r^2 - \bar{M}^2$$

Analogous relations hold for the averages of the angular distribution, with the substitution $2/\alpha_\Lambda = \varepsilon_j^i = 1$.

Quantities of physical interest can be deduced from (21) and (22). For example we know that the coefficients b_1, c_2, c_3 and d_i multiplied by I_0 are of the form $\sin\theta\ \Sigma_l a_l P_l(\cos\theta)$, so that a can be obtained taking $f(\theta) = \frac{1}{\sin\theta} P_l(\cos\theta)$.

Furthermore, if we are interested in averages, or ratios of averages, which are independent of θ, the weight function $f(\theta)$ remains arbitrary and can usefully be chosen to minimize the statistical error.

5 - TESTS FOR Ξ SPIN.

5.1 - We now derive some useful limitations on the coefficients $b_1(\theta)$, $c_i(\theta)$ and $d_i(\theta)$ for the spin $\frac{3}{2}$ case.

The diagonal elements of the density matrix (3) are the probabilities to find the Ξ in the various spin component states. Taking $\vec{n}$ as the direction of spin quantization, we get for the probability I_μ to have the Ξ with spin component μ:

$$I_\mu(\theta) = \frac{1}{4} + b_1(\theta)\frac{1}{\sqrt{5}}\mu - [c_1(\theta)+c_2(\theta)]\frac{1}{16}(4\mu^2-5) + \tag{23}$$
$$+ [d_1(\theta) - \tfrac{1}{2}d_2(\theta)]\frac{1}{12\sqrt{5}}(20\mu^3-41\mu) \qquad \left(-\tfrac{3}{2} \le \mu \le \tfrac{3}{2}\right)$$

Limitations on the coefficients are obtained by the conditions $I_\mu \geq 0$. For example we find

$$|b_1(\theta)| \le \frac{3}{2\sqrt{5}} \tag{24}$$

From (24) we have obvious limitations on the averages (13a), (13e), (14d) and (18a), which must be satisfied if the spin is $\frac{3}{2}$. In particular since $|\alpha| \leq 1$

$$\left| \langle \omega_1 \theta \rangle \right| = \frac{2}{3\sqrt{5}} \left| b_1(\theta) \alpha \right| \leq \frac{1}{5}$$

which is one of the Lee-Yang conditions.[4]

5.2 - The coefficients b_1, c_i and d_i are not independent, but are subject to a quadratic relation. In fact, as shown by Peshkin[5], we can write the density matrix of the Ξ in the form

$$\mathcal{U} = \frac{1}{2} |\psi\rangle\langle\psi| + \frac{1}{2} |R\psi\rangle\langle R\psi|$$

as a consequence of the fact that the initial beam in the original production process of Ξ is an incoherent mixture of two spin states, one with the nucleon spin parallel to the incident momentum, the other with the nucleon spin antiparallel to the incident momentum.

The two states transform into each other under the operation denoted by R, space inversion plus a rotation of π around the normal to the production plane. From parity conservation in the production process it follows that the statistical matrix for the produced Ξ consists of an incoherent superposition of two pure spin states, ψ and $R\psi$.

For J = $\frac{3}{2}$ we obtain

$$Tr\, \mathcal{U}^2 = \frac{1}{2} + \frac{1}{2} \left[\Sigma_\mu (-1)^{\frac{3}{2}-\mu} I_\mu \right]^2$$

From (3) and (23) we find that the following relation holds:

$$\frac{3}{5} b_1^2 + c_1^2 + c_2^2 - c_1 c_2 + \frac{3}{4} c_3^2 - \frac{3}{5} d_1^2 + \frac{4}{15} d_2^2 + \frac{3}{5} d_1 d_2 +$$

$$+ \frac{5}{12} d_3^2 - \frac{8}{5} b_1 d_1 + \frac{4}{5} b_1 d_2 = \frac{1}{4} \quad . \tag{26}$$

5.3 - From (23) and (26) we can deduce a more conclusive criterion for the spin of Ξ .

In fact not only we see that all the coefficients cannot be simultaneously zero, but also that b_1 cannot be different from zero alone. Indeed from (26) we would have $|b_1| = \frac{1}{2}\sqrt{5/3}$ whereas from (23): $|b_1| \leq \sqrt{5/6}$. These two conditions would contradict each other.

One has then the following criterion. If the coefficients c_i and d_i all turn out to be zero, the spin must be $\frac{1}{2}$.

5.4 - We finally notice that for θ = o or 180° the Adair criterion[6] must be applicable.

Since for θ = o or 180° the angular distribution $4\pi I(\vec{v})$, given by (4), reduces to $\frac{1}{2}[3(u\cdot v)^2+1]$, all the coefficients but c_1 must be zero, and $c_1 = -\frac{1}{2}$.

We then see, from (5) that the polarization $\underline{P}$ of the Λ is completely longitudinal, and its value is $+\bar{\alpha}$ (in the direction of the Λ momentum).

This result is valid for any spin value, as long as the production satisfies Adair's condition.

6.- S AND p WAVES - TESTS FOR Ξ PARITY.

6.1 - The discussion we have given so far has been completely general. In particular no simplifying assumptions of any kind have been made on the production reaction (1). For completeness we shall briefly list the particular formulas that one obtains for the coefficients of the production density matrix if only s and p waves are supposed to partecipate to the production reaction. The formulas that we report in this section are all readily obtainable from usual arguments (see for instance ref. 2). As long as reaction (1) does not take place at very high energy if will be a very useful hypothesis to suppose that only s and p waves are involved in the initial and final state. This hypothesis might in fact be valid up to rather high energy if one considers that the longest range of the interaction for reaction (1) will be that due to the exchange of two K-mesons. One this basis one can estimate that up to an energy as high as 1GeV in the c.m.s. it might still be a valid conjecture to suppose that only s and p waves contribute. The production angular distribution $I_o(\theta)$ is then of the form

$$I_o(\theta) = C(a + b\cos\theta + c\cos^2\theta) \tag{27}$$

where

$$C^{-1} = 4\pi(a + \tfrac{1}{3}c)$$

and the coefficients a, b, and c have different expressions according to the spin S and parity P of Ξ. We call $S^J_{l'l}$ the production matrix element for total angular momentum J, initial l-wave, and final l'-wave. For S = $\frac{1}{2}$ we have: a, b, and c are in general different from zero for P = + 1, while c = o for P = - 1.

Namely, for $S = \frac{1}{2}$ and $P = +1$,

$$a = |S_{00}^{\frac{1}{2}}|^2 + |S_{11}^{\frac{1}{2}}|^2 + |S_{11}^{\frac{3}{2}}|^2 - 2\,Re\,(S_{11}^{\frac{1}{2}} S_{11}^{\frac{3}{2}*})$$

$$b = 2\,Re\,(S_{00}^{\frac{1}{2}} S_{11}^{\frac{1}{2}*} + 2 S_{00}^{\frac{1}{2}} S_{11}^{\frac{3}{2}*})$$

$$c = 3|S_{11}^{\frac{3}{2}}|^2 + 6\,Re\,(S_{11}^{\frac{1}{2}} S_{11}^{\frac{3}{2}*})$$

For $S = \frac{1}{2}$ and $P = -1$,

$$a = |S_{01}^{\frac{1}{2}}|^2 + |S_{10}^{\frac{1}{2}}|^2$$

$$b = 2\,Re\,(S_{01}^{\frac{1}{2}} S_{10}^{\frac{1}{2}*})$$

$$c = 0$$

For $S = \frac{3}{2}$ we have: $b = 0$ for $P = +1$, and $c = 0$ for $P = -1$.

For $S = \frac{3}{2}$ and $P = +1$,

$$a = |S_{11}^{\frac{1}{2}}|^2 + \frac{14}{5}|S_{11}^{\frac{3}{2}}|^2 + \sqrt{\frac{2}{5}}\,Re\,(S_{11}^{\frac{1}{2}} S_{11}^{\frac{3}{2}*})$$

$$b = 0$$

$$c = -\frac{12}{5}|S_{11}^{\frac{3}{2}}|^2 - 3\sqrt{\frac{2}{5}}\,Re\,(S_{11}^{\frac{1}{2}} S_{11}^{\frac{3}{2}*})$$

For $S = \frac{3}{2}$ and $P = -1$

$$a = |S_{10}^{\frac{1}{2}}|^2 + 2|S_{01}^{\frac{3}{2}}|^2$$

$$b = -2\sqrt{2}\,Re\,(S_{10}^{\frac{1}{2}} S_{01}^{\frac{3}{2}*})$$

$$c = 0$$

For $S = \frac{1}{2}$ the polarization of the Ξ is described by a

vector directed along $\vec{n}$, whose component along $\vec{n}$ we call P_o.
With only S and p waves

$$I_o(\theta)\, P_o(\theta) = C \sin\theta \,(p + q\cos\theta) \tag{28}$$

For $S = \frac{1}{2}$ and $P = +1$

$$p = 2\,Im\left(S_{11}^{\frac{1}{2}} S_{00}^{\frac{1}{2}*} + S_{00}^{\frac{1}{2}} S_{11}^{\frac{3}{2}*}\right), \qquad q = 6\,Im\left(S_{11}^{\frac{1}{2}} S_{11}^{\frac{3}{2}*}\right)$$

For $S = \frac{1}{2}$ and $P = -1$

$$p = 2\,Im\left(S_{10}^{\frac{1}{2}} S_{01}^{\frac{1}{2}*}\right), \qquad q = 0$$

We notice that $q = o$ if $P = -1$.

For $S = \frac{3}{2}$ the coefficients b_1, c_i and d_i are given by

$$\begin{cases} I_o(\theta)\, b_1(\theta) = C \sin\theta\,(z_1 + z_2\cos\theta) \\ I_o(\theta)\, c_1(\theta) = C\,(z_3 + z_4\cos\theta + z_5\cos^2\theta) \\ I_o(\theta)\, c_2(\theta) = C\, z_6 \sin^2\theta \\ I_o(\theta)\, c_3(\theta) = C \sin\theta\,[z_4 + (z_5 + z_6)\cos\theta] \\ I_o(\theta)\, d_1(\theta) = 0 \end{cases}$$

$$z_2 = \frac{3}{5\sqrt{2}} \, Im \, (s_{11}^{\frac{1}{2}} s_{11}^{\frac{3}{2}*})$$

$$z_3 = \frac{3}{5} \, | s_{11}^{\frac{3}{2}} |^2 + 3\sqrt{\frac{2}{5}} \, Re \, (s_{11}^{\frac{1}{2}} s_{11}^{\frac{3}{2}*})$$

$$z_4 = 0$$

$$z_5 = -\frac{1}{2} \, | s_{11}^{\frac{1}{2}} |^2 - \frac{4}{5} \, | s_{11}^{\frac{3}{2}} |^2 - 2\sqrt{\frac{2}{5}} \, Re \, (s_{11}^{\frac{1}{2}} s_{11}^{\frac{3}{2}*})$$

$$z_6 = -\frac{1}{2} \, | s_{11}^{\frac{1}{2}} |^2 + \frac{8}{5} \, | s_{11}^{\frac{3}{2}} |^2 + \sqrt{\frac{2}{5}} \, Re \, (s_{11}^{\frac{1}{2}} s_{11}^{\frac{3}{2}*})$$

For $S = \frac{3}{2}$ and $P = -1$

$$z_1 = \sqrt{\frac{5}{2}} \, Im \, (s_{10}^{\frac{1}{2}} s_{01}^{\frac{3}{2}*})$$

$$z_2 = 0$$

$$z_3 = - | s_{01}^{\frac{3}{2}} |^2$$

$$z_4 = \sqrt{2} \, Re \, (s_{10}^{\frac{1}{2}} s_{01}^{\frac{3}{2}*})$$

$$z_5 = z_6 = -\frac{1}{2} \, | s_{10}^{\frac{1}{2}} |^2$$

We notice that $z_1 = z_4 = o$ if $P = + 1$, while $z_2 = o$ if $P = - 1$.

Once the spin of Ξ is known, from (27) and (28) or (29) one might possibly obtain indications on the Ξ parity.

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